matrix

University of Sawa
College of Engineering Technology

Department of Medical Devices

First Stage

Lect : Haider Abdl R Jassim

Lecture number : 5

What is a matrix?

Matrix:

- Mathematical objects with operations

Matrix in computer graphics:

- Defines a coordinate frame
- Defines a transformation
- Handy tool for manipulating transformations

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Matrix

Matrix: rectangular array of *elements*

Element: quantity (value, expression, function, …)

Examples:

$$\begin{bmatrix} 3.60 & -0.01 & 2.1 \\ -5.46 & 0.00 & 1.6 \end{bmatrix}, \quad \begin{bmatrix} e^x & x \\ e^{2x} & x^2 \end{bmatrix}, \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Matrix

$r \times c$ matrix: r rows, c columns

m_{ij} : element at row i and column j.

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1c} \\ m_{21} & m_{22} & \cdots & m_{2c} \\ \vdots & \vdots & & \vdots \\ m_{r1} & m_{r2} & \cdots & m_{rc} \end{bmatrix}$$

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Matrix

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$r \times c$ elements

Matrix

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c columns

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Matrix

$r \times c$ matrix: r rows, c columns

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$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1c} \\ m_{21} & m_{22} & \cdots & m_{2c} \\ \vdots & \vdots & & \vdots \\ m_{r1} & m_{r2} & \cdots & m_{rc} \end{bmatrix}$$

r rows

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Matrix

Column vector:
matrix with $c = 1$

$$\mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_r \end{bmatrix}$$

Used for vectors
(and points)

- *Row vector*:
matrix with $r = 1$

$$\mathbf{V} = \begin{bmatrix} v_1 & v_2 & \cdots & v_c \end{bmatrix}$$

Scalar:
matrix with $r = 1$ and $c = 1$

$$\mathbf{V} = \begin{bmatrix} v \end{bmatrix}$$

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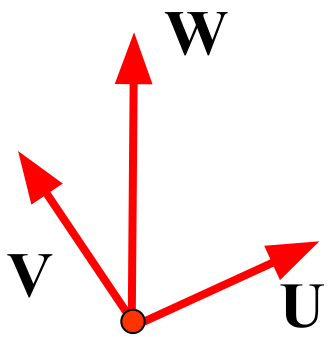
Matrix

Matrix as collection of column vectors:

$$\mathbf{M} = \begin{bmatrix} \mathbf{U} & \mathbf{V} & \mathbf{W} \end{bmatrix} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$

Matrix contains an axis-frame:



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Operations on matrices

- Multiplication with scalar
  - simple
- Addition
  - simple
- Matrix-matrix multiplication
  - More difficult, but the most important

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Scalar Matrix multiplication

Matrix **M** multiplied with scalar *s*:

$$\mathbf{A} = s\mathbf{M} = \begin{bmatrix} sm_{11} & sm_{12} & \cdots & sm_{1c} \\ sm_{21} & sm_{22} & \cdots & sm_{2c} \\ \vdots & \vdots & & \vdots \\ sm_{r1} & sm_{r2} & \cdots & sm_{rc} \end{bmatrix}, \text{ i.e., } a_{ij} = sm_{ij}$$

$$\text{example}: \begin{bmatrix} 3x & 12 \\ 6 & 2y \end{bmatrix} = 3\begin{bmatrix} x & 4 \\ 2 & y \end{bmatrix}$$

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Scalar Matrix addition

$$\mathbf{M} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1c} \\ b_{21} & b_{22} & \cdots & b_{2c} \\ \vdots & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rc} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1c}+b_{1c} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2c}+b_{2c} \\ \vdots & \vdots & & \vdots \\ a_{r1}+b_{r1} & a_{r2}+b_{r2} & \cdots & a_{rc}+b_{rc} \end{bmatrix}, \text{i.e., } m_{ij} = a_{ij} + b_{ij}$$

$$\text{example}: \begin{bmatrix} 3x & 12 \\ 6 & 2y \end{bmatrix} + \begin{bmatrix} 4 & -10 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 3x+4 & 2 \\ 8 & 2y \end{bmatrix}$$

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Scalar Matrix addition

$$\mathbf{M} = \mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1c} \\ b_{21} & b_{22} & \cdots & b_{2c} \\ \vdots & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rc} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1c}+b_{1c} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2c}+b_{2c} \\ \vdots & \vdots & & \vdots \\ a_{r1}+b_{r1} & a_{r2}+b_{r2} & \cdots & a_{rc}+b_{rc} \end{bmatrix}, \text{i.e.,}\ m_{ij} = a_{ij} + b_{ij}$$

- Just add elements pairwise
- A and B must have the same number of rows and columns
- Generalization of vector and scalar addition

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Matrix Matrix multiplication

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Matrix Matrix multiplication

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{mn} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{bmatrix}$$

$i = 2\ ;\ k \rightarrow$ $j = 1\ ;\ k \downarrow$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

- c_{ij}: dot product of *row* vector a_i and *column* vector b_j
- #columns **A** must be the same as #rows **B**: $n = p$
- **C**: $m \times q$ matrix

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Matrix Matrix multiplication

Example:

$$\begin{bmatrix} 2 & 3 \\ -1 & -4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2\times1+3\times3 & 2\times2+3\times4 \\ -1\times1-4\times3 & -1\times2-4\times4 \\ 0\times1+5\times3 & 0\times2+5\times4 \end{bmatrix} = \begin{bmatrix} 11 & 16 \\ -13 & -16 \\ 15 & 20 \end{bmatrix}$$

$i = 3$ $j = 2$ $i=3, j=2$

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Matrix Matrix multiplication

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{AB} = ?$$

$$\mathbf{BA} = ?$$

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Matrix Matrix multiplication

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + 2 \times 5 + 3 \times 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

$$\mathbf{BA} = ?$$

Matrix Matrix multiplication

Example:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

AB ≠ BA

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 4 + 2 \times 5 + 3 \times 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 \times 1 & 4 \times 2 & 4 \times 3 \\ 5 \times 1 & 5 \times 2 & 5 \times 3 \\ 6 \times 1 & 6 \times 2 & 6 \times 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

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Matrix Matrix multiplication

- **AB ≠ BA**

  Matrix matrix multiplication is not commutative!

  Order matters!

- **A(B+C) = AB+AC**

  Matrix matrix multiplication is distributive

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Example transformation sequence (coordinate version)

$$X' \leftarrow X: \quad x' = px + qy, \quad y' = rx + sy;$$

$$X'' \leftarrow X': \quad x'' = ax' + by', \quad y'' = cx' + dy'.$$

Then $X'' \leftarrow X$:

$$x'' = a(px + qy) + b(rx + sy),$$

$$y'' = c(px + qy) + d(rx + sy);$$

or

$$x'' = (ap + br)x + (aq + bs)y,$$

$$y'' = (cp + dr)x + (cq + ds)y.$$

Unclear!

Error-prone!

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Example transformation sequence (matrix vector version)

$$X' \leftarrow X: \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix};$$

$$X'' \leftarrow X': \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Then $X'' \leftarrow X$:

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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Example transformation sequence (compact matrix vector version)

$X' \leftarrow X: \quad \mathbf{X'} = \mathbf{AX};$

$X'' \leftarrow X': \quad \mathbf{X''} = \mathbf{BX'}$

Here some more detail:

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{X''} = \begin{bmatrix} x' \\ y' \end{bmatrix}, \mathbf{X''} = \begin{bmatrix} x'' \\ y'' \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}; \mathbf{B} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then $X'' \leftarrow X$:

$$\mathbf{X''} = \mathbf{BAX}$$

Matrix vector notation allows for compactness and genericity!

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Matrix Transpose

$$\mathbf{M} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2c} \\ \vdots & \vdots & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rc} \end{bmatrix}$$

Transpose matrix:

Interchange rows and columns.

$r \times c$ matrix $\mathbf{M} \rightarrow c \times r$ matrix $\mathbf{M}^{\mathrm{T}}$

$$\mathbf{M}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{r1} \\ a_{12} & a_{22} & \cdots & a_{r2} \\ \vdots & \vdots & & \vdots \\ a_{1c} & a_{2c} & \cdots & a_{rc} \end{bmatrix}, \text{ i.e., } m^T_{ji} = m_{ij}$$

$$\text{examples}: \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad [x \quad y \quad z]^T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Matrix Inverse

Simple algebra puzzle. Let $ax = b$. What is x?

$$
\begin{aligned}
ax &= b && \text{Multiply left and right with } a^{-1} \\
a^{-1}ax &= a^{-1}b && \text{Use } a^{-1}a \text{ equals } 1 \\
x &= a^{-1}b && \text{Done}
\end{aligned}
$$

Linear algebra puzzle. Let **MU = V**. What is **U**?

$$
\begin{aligned}
\mathbf{MU} &= \mathbf{V} && \text{Multiply left and right with } \mathbf{M}^{-1} \\
\mathbf{M}^{-1}\mathbf{MU} &= \mathbf{M}^{-1}\mathbf{V} && \text{Use } \mathbf{M}^{-1}\mathbf{M} \text{ equals } \mathbf{I} \\
\mathbf{U} &= \mathbf{M}^{-1}\mathbf{V} && \text{Done!}
\end{aligned}
$$

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Matrix Inverse

Inverse of a $n \times n$ (square) matrix $\mathbf{M}$ is denoted $\mathbf{M}^{-1}$, with

$\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$,

where $\mathbf{I}$ is the identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, \text{ i.e., } \mathbf{I}_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

$\mathbf{I}$ is a *diagonal matrix*, with all 1's on the diagonal.

$\mathbf{M}^{-1}$ "undoes" the effect of $\mathbf{M}$,

multiplying with $\mathbf{I}$ has no effect.

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Matrix Inverse examples

$$[4]^{-1} = [1/4]$$

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -5/2 & 3/2 \end{bmatrix} = \begin{bmatrix} 3\times 2 - 2\times 5/2 & -3\times 1 + 2\times 3/2 \\ 5\times 2 - 4\times 5/2 & -5\times 1 + 4\times 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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Matrix Inverse examples

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \; ?$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & -ab + ba \\ -ab + ba & -cb + ad \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}^{-1} ?$$ Does not exist, cannot be inverted.

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Matrix Inverse

- Does not always exist
- In general: if *determinant* = | **M** | = 0, matrix cannot be inverted
- Inverse for n = 1 and n = 2: easy
- Inverse for higher n: use library function
- Important special case: *orthonormal matrix*.

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Overview

- Coordinates, points, vectors
  - definitions, operations, examples
- Matrices
  - definitions, operations, examples