SAWA UNIVERSITY

SECOND ORDER LINEAR EQUATION WITH CONSTANT COEFFICIENTS

Sawa University

Engineering Technology

Department of Refrigeration and Air-Conditioning Engineering

$3^{rd} Year$

Lecture 4

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Second order linear equation with constant coefficients

A second order O.D.E is called linear if it can be written:

$\boldsymbol{F(x) \; = \; y'' + P(x)y' + Q(x)y}$

- If f(x) = 0 → Homogeneous (Homo)
- If f(x)≠ 0 → Non-homogeneous (Non Homo)

For example

$\boldsymbol{y'' + 5y' + 3y = 0}$ $\qquad$ $\boldsymbol{Homo.}$

$\boldsymbol{y'' + 5y' = e^{-x}cosx}$ $\qquad$ $\boldsymbol{Non\ \ Homo.}$

1- Homogeneous Linear D.E with constant coefficient

$y'' + a.y' + b.y = 0 \qquad ,a,b = constant$

Solution

1) Find characteristic equation:

Let $y' \; = D \qquad\qquad y'' \; = D^2$

2) $y = e^{Dx}$

EX1:

$\boldsymbol{y'' + 3y' - 4y = 0}$

$D^2 + \boldsymbol{3D - 4 = 0}$

$$D = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1, \quad b = 3, \quad c = -4$

$$D = \frac{-3 \pm \sqrt{3^2 - 4(1)(-4)}}{2(1)}$$

$$D = \frac{-3 \pm \sqrt{9+16}}{2}$$

$$D = \frac{-3 \pm \sqrt{25}}{2}$$

$$D = \frac{-3 \pm 5}{2}$$

- First root:

$$D_1 = \frac{-3+5}{2} = \frac{2}{2} = 1$$

- Second root:

$$D_2 = \frac{-3-5}{2} = \frac{-8}{2} = -4$$

There are three cases in solving D.E

Case I: Has Two real roots (different)

Solution:

G.S $\quad$ y = $C1\, e^{D1x} + C2\, e^{D2x}$

EX2: $\boldsymbol{solve \quad D.E. \quad y'' - 4y = 0}$

$y' = D$

$D^2 - 4 = 0 \rightarrow D^2 = 4 \rightarrow$ D= $\pm 2 \quad \rightarrow \quad D1 = +2 \;, \quad D2 = -2$

G.S $\quad$ y = $C1\, e^{2x} + C2\, e^{-2x}$

Case II : Has same double roots

$D1 = D2$

Solution:

G.S y = $C1\ e^{D1x} + \ C2\ e^{D1x}\ * x$

y = $C1\ e^{D1x} + \ C2\ x\ e^{D1x}$

EX3: Find the general solution of y''+8y'+16y=0

$m^2 + 8m + 16 = 0$

$(m+4)(m+4)=0$

$m1=m2=-4$

y = $C1\ e^{-4x} + \ C2\ x\ e^{-4x}$

Case III: Has two complex roots

D = $a \pm B\sqrt{-1}$

D = $a \pm B\ i$

G.S y = $C1\ e^{ax} sinB(x) + \ C2\ e^{ax} cosB(x)$

EX4: Solve D.E

$y'' + 2y' + 5y = 0$

$D^2 + 2D + 5 = 0$

$$D = -b \ \pm \frac{\sqrt{b^2 - 4AC}}{2A}$$

a=1,b=2,c=5

$$D = \frac{-2 \pm \sqrt{-16}}{2}$$

$$\sqrt{-16} = 4j$$

$$D = \frac{-2 \pm 4j}{2} = -1 \pm 2j$$

D = $-1 \pm 2\sqrt{-1}$

D = $-1 \pm 2i$

G.S y = $C1\ e^{ax} sinB(x) + \ C2\ e^{ax} cosB(x)$

G.S y = $C1\ e^{-x} sin2(x) + \ C2\ e^{-x} cos2(x)$

Application 1: Damped Vibrations

$$\boxed{m \frac{d^2x}{dt^2} + c\,\frac{dx}{dt} + kx = 0}$$

حيث:

- m: الكتلة (mass)
- c: معامل التخميد (damping coefficient)
- k: ثابت النابض (spring constant)

EX5: A mass–spring–damper system consists of a mass m= 2kg, a damping coefficient c=8 N·s/m, and a spring with stiffness k=32 N/m.If the mass is displaced and released, determine the displacement x(t) as a function of time.

Sol:

$$\boxed{m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0}$$

Given the differential equation for a mass–spring–damper system:

$2x'' + 8x' + 32x = 0$

Dividing through by 2:

$x'' + 4x' + 16x = 0$

$r^2 + 4r + 16 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here:

$a = 1,\ b = 4,\ c = 16$

$$r = \frac{-4 \pm \sqrt{-48}}{2}$$

$$\sqrt{-48} = \sqrt{48}\,j = 4\sqrt{3}\,j$$

$$r = \frac{-4 \pm 4\sqrt{3}j}{2}$$

$$r = -2 \pm 2\sqrt{3}j$$

:

$$\boxed{r = -2 \pm 2\sqrt{3}j}$$

General solution:

$$x(t) = e^{-2t}\left(C_1 \cos(2\sqrt{3}t) + C_2 \sin(2\sqrt{3}t)\right)$$

Home work:

For each of the following problem find the general solution of the differential equation:

$$y'' + 2y' + 2y = 0$$

$$y'' - 6y' + 9y = 0$$

$$y'' + 2y' - 4y = 0 \quad ; \quad y'(0) = -6 \text{ and } y(0) = 6$$

Thank You