SAWA UNIVERSITY

GAMMA FUNCTION

Sawa University

Engineering Technology

Department of Refrigeration and Air-Conditioning Engineering

$3^{rd} Year$

Lecture 9

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Lecture 9

Gamma Function

Let (x) be a real number that is not zero or a negative integer. Then the gamma function that denoted by(Γ), is defined as:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \ldots\ldots\ldots (1)$$

Note that the above integral is not defined for all values of x. In fact we can only say that $\Gamma(x)$ is defined everywhere except at (0) and for negative integers.

❖ **Properties of gamma function**

1- $\Gamma(x+1) = x\Gamma(x)$

Proof:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Use integration by parts to obtain

$$\int v\, du = uv - \int u\, dv.$$

$$u = t^x, \quad dv = e^{-t} dt \quad \Longrightarrow \quad du = x t^{x-1} dt, \; v = -e^{-t},$$

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\, dt = [-t^x e^{-t}]_0^\infty + \int_0^\infty x t^{x-1} e^{-t}\, dt$$

The boundary term is zero (since $t^x e^{-t} \to 0$ as $t \to 0$ and $t \to \infty$).

$$\Gamma(x+1) = x \int_0^\infty t^{x-1} e^{-t}\, dt = x\,\Gamma(x)$$

Note:

$$\Gamma(n+1) = n\Gamma(n)$$
$$\Gamma(n+2) = (n+1)\Gamma(n+1)$$
$$\Gamma(n) = (n-1)\Gamma(n-1)$$

2- $\boldsymbol{\Gamma}(\mathbf{1}) = 1$

Proof:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Substitute $x = 1$:

$$\Gamma(1) = \int_0^\infty t^{1-1} e^{-t}\, dt = \int_0^\infty e^{-t}\, dt$$

Now integrate:

$$\int e^{-t}\, dt = -e^{-t} + C$$

Therefore,

$$\Gamma(1) = \Big[-e^{-t}\Big]_0^\infty$$

$$= (-e^{-t})\Big|_0^\infty = (0) - (-1) = 1$$

$$\boxed{\Gamma(1) = 1}$$

3- $\boldsymbol{\Gamma}\left(\frac{\mathbf{1}}{\mathbf{2}}\right) = \sqrt{\pi}$

4- $\Gamma(n) = (n-1)!$

Proof:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

$$\Gamma(x+1) = x\,\Gamma(x)$$

Take $n = 1$:

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1 = 0!$$

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1!$$

$$\Gamma(3) = 2 \cdot \Gamma(2) = 2!$$

$$\Gamma(4) = 3 \cdot \Gamma(3) = 3!$$

$$\Gamma(n) = (n-1)!$$

Ex1: Evaluate $\Gamma(8)$

Sol:

$$\Gamma(n) = (n-1)!$$

$$\Gamma(8) = (8-1)! = 7!$$

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$\Gamma(8)=5040$

Ex2: Evaluate $\frac{\Gamma(6)}{2\,\Gamma(3)}$

Sol:

$$\Gamma(n) = (n-1)!$$

$$\frac{\Gamma(6)}{2\,\Gamma(3)} = \frac{(6-1)!}{2\cdot(3-1)!} = \frac{5!}{2\cdot 2!}$$

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120, \quad 2! = 2$$

So:

$$\frac{5!}{2\cdot 2!} = \frac{120}{2\cdot 2} = \frac{120}{4} = 30$$

Ex3: Evaluate

$$\frac{\Gamma(7)}{3\Gamma(4)}$$

$$\Gamma(7) = 6! = 720, \quad \Gamma(4) = 3! = 6$$

$$\frac{720}{3\times 6} = \frac{720}{18} = 40$$

$$\frac{\Gamma(7)}{3\Gamma(4)} = 40$$

Ex4: Using definition $\Gamma(5)$

Sol:

$$\Gamma(5) = \int_0^\infty t^4 e^{-t} dt$$

$$\Gamma(5) = 4\Gamma(4) = 4 \cdot 3\Gamma(3) = 4 \cdot 3 \cdot 2\Gamma(2) = 24 \cdot 1 = 24$$

Ex5: Evaluate $\int_0^\infty x^3 e^{-x}\, dx$

Sol:

By compression with the general Gamma equation

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Then we find that

$$\Gamma(4) = \int_0^\infty x^{4-1} e^{-x} dx$$

$$= \int_0^\infty x^3 e^{-x} dx$$

$$= 3!$$

Ex6: $\Gamma \frac{3}{2}$

$$\Gamma(x+1) = x\Gamma(x)$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

Ex7: $\Gamma \frac{5}{2}$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

H.w: Evaluate:

1- $\Gamma \frac{7}{2}$

2- $\frac{\Gamma(8)}{4\,\Gamma(5)}$

Thank You