SAWA UNIVERSITY

THE LAPLACE TRANSFORM

Sawa University

Engineering Technology

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Lecture 10

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Lecture 10

The Laplace Transform

The method of Laplace transforms is a system that relies on algebra (rather than calculus-based methods) to solve linear differential equations. While it might seem to be a somewhat cumbersome method at times, it is a very powerful tool that enables us to readily deal with linear differential equations with discontinuous forcing functions.

طريقة تحويل لابلاس (Laplace Transform) هي أسلوب رياضي لحل المعادلات التفاضلية الخطية بدلاً من استخدام طرق التكامل أو التفاضل المباشرة (الطرق التقليدية في التفاضل والتكامل)، تعتمد هذه الطريقة على العمليات الجبرية مثل الجمع، الطرح، الضرب، والقسمة في متغير جديد يسمى s-domain
يعني أننا نحول المعادلة من الشكل الزمني (t-domain) إلى الشكل الترددي أو المركب (s-domain) ، فنحوّل التفاضل إلى عملية ضرب بسيطة بـ s، وهكذا يصبح الحل أبسط بكثير

Definition: Let $f(t)$ be defined for $t \geq 0$. The Laplace transform of $f(t)$ denoted by $F(s)$ or $L\{f(t)\}$, is an integral transform given by the Laplace integral

$$\boxed{L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt}$$

Example 1: Find the Laplace transform of $f(t) = c$, where c is a constant.

Solution:

$$L\{c\} = F(s) = \int_0^\infty ce^{-st}dt = \frac{-c}{s}\, e^{-st}\Big|_0^\infty$$

$$L\{c\} = F(s) = \frac{-c}{s}(0-1) = \frac{c}{s}$$

Example 2: Find the Laplace transform of $L\{1\}$

Solution:

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}dt$$

$$= \lim_{b\to\infty}\frac{-e^{-st}}{s}\Big|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}$$

Table of Laplace Transforms

$f(t)$	$F(s) = L\{f(t)\}$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$

Example 3: Find the Laplace transform of:

1. $f(t) = e^{-2t} + t^3 - 4.$
2. $f(t) = \sin 2t - \cos 2t.$

Solution:

1. $L\{e^{-2t} + t^3 - 4\} = L\{e^{-2t}\} + L\{t^3\} - L\{4\}$

$$F(s) = \frac{1}{s+2} + \frac{\Gamma(3+1)}{s^{3+1}} - \frac{4}{s}$$

$$F(s) = \frac{1}{s+2} + \frac{3!}{s^4} - \frac{4}{s} = \frac{1}{s+2} + \frac{6}{s^4} - \frac{4}{s}$$

2. $L\{\sin 2t - \cos 2t\} = L\{\sin 2t\} - L\{\cos 2t\}$

$$F(s) = \frac{2}{s^2+4} - \frac{s}{s^2+4}$$

Example 4:

Solution:

$$f(t) = e^{-3t} + \sin(2t) - \frac{1}{2}$$

- $\mathcal{L}\{e^{-3t}\} = \dfrac{1}{s+3}$
- $\mathcal{L}\{\sin(2t)\} = \dfrac{2}{s^2+4}$
- $\mathcal{L}\{-\tfrac{1}{2}\} = -\tfrac{1}{2} \cdot \dfrac{1}{s} = -\dfrac{1}{2s}$

$$\boxed{F(s) = \frac{1}{s+3} + \frac{2}{s^2+4} - \frac{1}{2s}}$$

Example 5:

Solution:

$$f(t) = 4\cos(3t) + t$$

- $\mathcal{L}\{4\cos(3t)\} = 4 \cdot \dfrac{s}{s^2+9} = \dfrac{4s}{s^2+9}$
- $\mathcal{L}\{t\} = \dfrac{1}{s^2}$

$$\boxed{F(s) = \frac{4s}{s^2+9} + \frac{1}{s^2}}$$

Some Properties of the Laplace Transform

1. If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n F^{(n)}(s)$.

2. If $L\{f(t)\} = F(s)$, then $L\{e^{at} f(t)\} = F(s-a)$

Example 6: Find Laplace Transform of

1. $f(t) = t^2 e^{4t}$ 2. $f(t) = e^{-t}\sin 2t$

Solution:

1. $$L\{e^{4t}\} = \frac{1}{s-4}$$

$$L\{t^2 e^{4t}\} = (-1)^2 F''(s) = \frac{d^2}{ds^2}\left(\frac{1}{s-4}\right) \quad \text{(by property 1)}$$

$$= \frac{d}{ds}\left(\frac{-1}{(s-4)^2}\right) = \frac{2}{(s-4)^3}$$

We can solve this example by using property 2 as follow:

$$L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

$$L\{t^2 e^{4t}\} = \frac{2}{(s-4)^3}$$

2. $$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{e^{-t}\sin 2t\} = \frac{2}{(s+1)^2+4} \quad \text{(by property 2)}$$

Example 7: Find Laplace Transform of

Find $L\{te^{-2t}\}$

$$L\{e^{-2t}\} = \frac{1}{s+2}$$

So $F(s) = \frac{1}{s+2}$

Now since we have t^1, we take **one derivative**, and multiply by $(-1)^1 = -1$:

$$L\{te^{-2t}\} = -\frac{d}{ds}\left(\frac{1}{s+2}\right)$$

$$-\frac{d}{ds}\left(\frac{1}{s+2}\right) = -(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}$$

Example 8: Find Laplace Transform of

$L\{t^2e^{-5t}\}$

Step 1: Laplace of t^2:

$$L\{t^2\} = \frac{2}{s^3}$$

Step 2: Multiply by e^{-5t} → shift $s \to s+5$:

$$L\{t^2e^{-5t}\} = \frac{2}{(s+5)^3}$$

Inverse Laplace Transform

In this lecture we look at the problem of finding inverse Laplace transforms. In other words, given $F(s)$, how do we find $f(t)$ so that $F(s) = L\{f(t)\}$.

Definition: The inverse Laplace transform of $F(s)$, denoted by $L^{-1}\{F(s)\}$ is the function $f(t)$ defined for $t \geq 0$ and satisfies $L\{f(t)\} = F(s)$.

Example 9: Find the Inverse Laplace Transform of

1. $F(s) = \dfrac{1}{s+2}$ 2. $F(s) = \dfrac{s+2}{s^2+4}$ 3. $F(s) = \dfrac{s+1}{s^2+s-12}$

Solution:

1. $L^{-1}\left\{\dfrac{1}{s+2}\right\} = e^{-2t}$

2. $L^{-1}\left\{\dfrac{s+2}{s^2+4}\right\} = L^{-1}\left\{\dfrac{s}{s^2+4} + \dfrac{2}{s^2+4}\right\}$

$$= L^{-1}\left\{\frac{s}{s^2+4}\right\} + L^{-1}\left\{\frac{2}{s^2+4}\right\} = \cos 2t + \sin 2t$$

3.

$$F(s) = \frac{s+1}{s^2+s-12}$$

$$s^2 + s - 12 = (s+4)(s-3)$$

$$F(s) = \frac{s+1}{(s+4)(s-3)}$$

$$\frac{s+1}{(s+4)(s-3)} = \frac{A}{s+4} + \frac{B}{s-3}$$

$$s+1 = A(s-3) + B(s+4)$$

$$s+1 = As - 3A + Bs + 4B$$

$$s+1 = (A+B)s + (-3A+4B)$$

$$\begin{cases} A+B=1 \\ -3A+4B=1 \end{cases}$$

From the first equation: $A = 1 - B$

Substitute into the second:

$$-3(1-B) + 4B = 1$$

$$-3 + 3B + 4B = 1$$

$$7B = 4 \Rightarrow B = \frac{4}{7}$$

Then:

$$A = 1 - \frac{4}{7} = \frac{3}{7}$$

$$F(s) = \frac{3/7}{s+4} + \frac{4/7}{s-3}$$

$$F(s) = \frac{3}{7}\frac{1}{s+4} + \frac{4}{7}\frac{1}{s-3}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$$

$$f(t) = \frac{3}{7}e^{-4t} + \frac{4}{7}e^{3t}$$

Example 10: Find the Inverse Laplace Transform of

$$L^{-1}\left\{\frac{12}{s^2+4s+13}\right\}$$

Solution:

$$L^{-1}\left\{\frac{12}{s^2+4s+13}\right\}=L^{-1}\left\{\frac{12}{s^2+4s+4+9}\right\}=L^{-1}\left\{\frac{4\times 3}{(s+2)^2+9}\right\}$$

$$=4L^{-1}\left\{\frac{3}{(s+2)^2+9}\right\}=4e^{-2t}\sin 3t$$

NOTE: ملاحظة

$$\mathcal{L}^{-1}\left\{\frac{a}{(s+b)^2+a^2}\right\}=e^{-bt}\sin(at)$$

Example 11: Find the Inverse Laplace Transform of

$$L^{-1}\left\{\frac{1}{s^3+s}\right\}$$

Solution:

$$\frac{1}{s^3+s}=\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{(s^2+1)}=\frac{As^2+A+Bs^2+Cs}{s(s^2+1)}$$

$$(A+B)s^2+Cs+A=1$$

$$A=1\ ,\qquad C=0\ \text{ and }\ A+B=0 \Rightarrow\ B=-1$$

$$L^{-1}\left\{\frac{1}{s^3+s}\right\}=L^{-1}\left\{\frac{1}{s}\right\}-L^{-1}\left\{\frac{s}{s^2+1}\right\}=1-\cos t$$

NOTE:

$F(s)$ Form	Numerator	Time-Domain Function $f(t)$
$\frac{k}{s+a}$	Constant	ke^{-at}
$\frac{s}{s^2+a^2}$	s	$\cos(at)$
$\frac{a}{s^2+a^2}$	Constant = a	$\sin(at)$
$\frac{s+b}{(s+b)^2+a^2}$	$s+b$	$e^{-bt}\cos(at)$
$\frac{a}{(s+b)^2+a^2}$	Constant = a	$e^{-bt}\sin(at)$
$\frac{n!}{s^{n+1}}$	$n!$	t^n
$\frac{1}{(s+a)^n}$	1	$\frac{t^{n-1}}{(n-1)!}e^{-at}$

H.w.1: Find Laplace Transform of

1- $L\{t^2 e^{3t}\}$

2- $L\{e^{-2t}\cos(3t)\}$

3- f(t) = $t^4 - 5e^{-t} + \sin(3t)$

H.w.2: Find Inverse Laplace Transform L^{-1} of

1- $\frac{s-2}{(s-2)^2+9}$

2- $\frac{2}{(s-3)^3}$

Thank You