SAWA UNIVERSITY

PARTIAL DIFFERENTIAL EQUATION (P.D.E)

Sawa University

Engineering Technology

Department of Refrigeration and Air-Conditioning Engineering

$3^{rd} Year$

Lecture 12

M.Sc. Rand Zwayen

Lecture 12

Partial Differential Equation (P.D.E)

A differential equation involving partial derivatives of a dependent variable (one or more) with more than one independent variable is called a partial differential equation, hereafter denoted as PDE. For example, the following equations are PDEs:

$$a\frac{\partial u}{\partial t}+b\frac{\partial u}{\partial x}=f(x,t) \quad \equiv \quad au_t+bu_x=f(x,t)$$

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0 \quad \equiv \quad u_{tt}-c^2u_{xx}=0 \qquad \text{(Wave Equation)}$$

$$\frac{\partial u}{\partial t}-T\frac{\partial^2 u}{\partial x^2}=0 \quad \equiv \quad u_t-Tu_{xx}=0 \qquad \text{(Heat Equation)}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \equiv \quad u_{xx}+u_{yy}=0 \qquad \text{(Laplace Equation)}$$

Partial differential equations , like ordinary diff. eq. are classified as either Linear or Non- linear

The linear second-order (P.D.E) is:

$$\boldsymbol{A\frac{d^2u}{dx^2}+B\frac{d^2u}{dxdy}+C\frac{d^2u}{dy^2}+D\frac{du}{dx}+E\frac{du}{dy}+Fu=G(x,y)}$$

$$\boldsymbol{A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=G(x,y)}$$

Classification of Linear Second-Order PDEs

If G(x,y) =0 , Then this equation is homogenous other wise, it is non homogenous.

$$\boldsymbol{\Delta=B^2-4AC}$$

This equation is called :

Elliptic If $\boldsymbol{B^2-4AC}<0$

Parabolic If $\boldsymbol{B^2-4AC}=0$

Hyperbolic If $\boldsymbol{B^2-4AC}>0$

EX1: Classify the following equations:

1- $3\frac{d^2u}{dx^2} = \frac{du}{dy}$

$3\frac{d^2u}{dx^2} - \frac{du}{dy} = 0$

A=3 B = C=0

$B^2 - 4AC$=0 the eq. is Parabolic

2- $\frac{d^2u}{dx^2} = \frac{d^2u}{dy^2}$

$$\frac{d^2u}{dx^2} - \frac{d^2u}{dy^2} = 0$$

A=1 B=0 C=-1

$B^2 - 4AC$= 0+4 > 0

the eq. is Hyperbolic

3- $\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

A= C=1 B=0

$B^2 - 4AC$= 0 - 4 < 0

the eq. is Elliptic

In practical problems, the following types of equation are generally used:

1. One – dimensional heat flow :

$$\frac{du}{dy} = C^2 \frac{d^2u}{dx^2}$$

Parabolic

2. One – dimensional wave equation :

$$\frac{d^2u}{dt^2} = C^2 \frac{d^2u}{dx^2}$$

Hyperbolic

3. Two– dimensional heat flow (Laplace equation) :

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$$

Elliptic

Solution of a P.D.E:

A solution of a linear partial differential equation

$$\boldsymbol{A\frac{d^2u}{dx^2} + B\frac{d^2u}{dxdy} + C\frac{d^2u}{dy^2} + D\frac{du}{dx} + \ldots\ldots\ldots}$$

is a function u (x,y) of two independent Variables

Method of separation variables:

In this method, we assume that the dependent variable is the product of two function each of which involves only one of the independent variable So two ordinary differential equations are formed.

$$\mathbf{u}_{(x,y)} = \mathbf{X}_{(x)} \;.\; \mathbf{Y}_{(y)}$$

to reduce a linear P.D.E in to two variables to O.D.E

$$\frac{du}{dx} = u_x = X'\,y$$

$$\frac{d^2u}{dx^2} = u_{xx} = X''\,y$$

$$\frac{du}{dy} = u_y = X\,y'$$

$$\frac{d^2u}{dy^2} = u_{yy} = Xy''$$

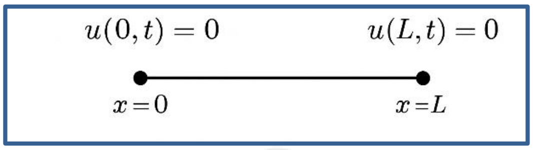
EX2: $\frac{d^2u}{dt^2} = C^2\,\frac{d^2u}{dx^2}$

$$T''\,X = C^2\,X''\,T$$

$$\frac{T''}{C^2T} = \frac{X''}{X}$$

Boundary Conditions B.C

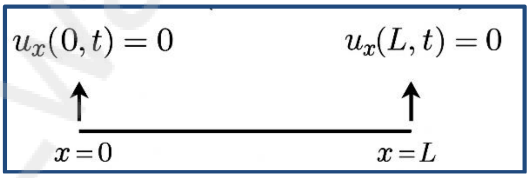
These are the values given at the edges of the domain:



- (fixed value):

$$u(0,t) = 0, \qquad u(L,t) = 0$$

- (derivative value):



$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

Used to determine the form of X(x) in the final solution

Initial Conditions I.C

These are the state of the system at t=0:

- Initial displacement:

$$u(x,0) = f(x)$$

EX3: $3\frac{du}{dx} + 2\frac{du}{dy} = 0 \qquad u(x,0)=4e^{-x}$

Sol:

Let $u_{(x,y)} = X_{(x)} \;.\; Y_{(y)}$

Where X is a function of x only and y is afunction of y only.

$$3\acute{X}Y + 2X\acute{Y} = 0 \;\rightarrow\; 3\acute{X}Y = -2\acute{Y}X$$

$$\frac{3\acute{X}}{X} = \frac{-2\acute{Y}}{Y} = \text{Constant} = \text{k}$$

$$\frac{3\acute{X}}{X} = \text{k} \quad \text{but} \quad \acute{X} = \frac{dX}{dx}$$

$$\frac{3\,dX}{x\,dx} = k \;\rightarrow\; \frac{3\,dX}{x} = k\,dx \;\rightarrow 3\int\frac{dX}{x} = \int k\,dx$$

$$3\ln X = kx + C1 \;\rightarrow\; \ln X = \frac{kx+C1}{3}$$

$$X = e^{\frac{kx}{3}+\frac{C1}{3}} \;\rightarrow X = e^{\frac{kx}{3}}\,.\,e^{\frac{C1}{3}} \rightarrow X = Ae^{\frac{kx}{3}}$$

$$\frac{-2\acute{Y}}{Y} = \text{k} \quad \text{but} \quad \acute{Y} = \frac{dY}{dy}$$

$$\frac{-2\,dY}{Y\,dy} = k \quad \rightarrow \int\frac{dY}{Y} = \int\frac{-k}{2}\,dy$$

$$\ln Y = \frac{-k}{2}\,y + C2 \;\rightarrow Y = e^{\frac{-k}{2}y+C2}$$

$$Y = e^{\frac{-k}{2}y}\,.\,e^{C2} \rightarrow Y = Be^{\frac{-ky}{2}}$$

$u_{(x,y)} = X_{(x)} \;.\; Y_{(y)}$

$u_{(x,y)} = Ae^{\frac{kx}{3}}.Be^{\frac{-ky}{2}}$

Let C=A.B

$u_{(x,y)} = C\, e^{\frac{k}{3}x - \frac{k}{2}y}$

B.C → $u(x,0)=4e^{-x}$ → x=x & y=0

$u(x,0)=4e^{-x} = C\, e^{\frac{k}{3}x - 0} = C\, e^{\frac{k}{3}x}$

$4e^{-x} = C\, e^{\frac{k}{3}x}$ من التشابه

$C = 4$ & $e^{-x} = e^{\frac{k}{3}x}$

$-1=\frac{k}{3}$ → $\boxed{k=-3}$

$u_{(x,y)} = C\, e^{\frac{k}{3}x - \frac{k}{2}y}$

$u_{(x,y)} = C\, e^{(\frac{-3}{3}x - \frac{(-3)}{2}y)}$

$$\boxed{u_{(x,y)} = C\, e^{-x + \frac{3}{2}y}}$$

EX4: $2\frac{du}{dx}+\frac{du}{dy}=0 \qquad u(x,0)=e^{2x}$

Sol:

Let $u_{(x,y)} = X_{(x)} \,.\, Y_{(y)}$

Where X is a function of x only and y is afunction of y only.

$$2\acute{X}Y + X\acute{Y} = 0 \;\rightarrow\; 2\acute{X}Y = -\acute{Y}X$$

$$\frac{2\acute{X}}{X} = \frac{-\acute{Y}}{Y} = \text{Constant} = \text{k}$$

$$\frac{2\acute{X}}{X} = \text{k} \quad \text{but} \quad \acute{X} = \frac{dX}{dx}$$

$$\frac{2\,dX}{X\,dx} = k \;\rightarrow\; \frac{dX}{X} = \frac{k}{2}\,dx \;\rightarrow\; \int\frac{dX}{x} = \int\frac{k}{2}\,dx$$

$$\ln X = \frac{k}{2}x + C1 \;\rightarrow$$

$$X = e^{\frac{kx}{2}+C1} \rightarrow X = e^{\frac{kx}{2}}.e^{C1} \rightarrow X = Ae^{\frac{k}{2}x}$$

$$\frac{-\acute{Y}}{Y} = \text{k} \quad \text{but} \quad \acute{Y} = \frac{dY}{dy}$$

$$\frac{-\,dY}{Y\,dy} = k \quad \rightarrow \int\frac{dY}{Y} = \int -k\,dy$$

$$\ln Y = -k\,y + C2 \rightarrow Y = e^{-ky+C2}$$

$$Y = e^{-ky}\,.\,e^{C2} \rightarrow Y = Be^{-ky}$$

$u_{(x,y)} = X_{(x)} \,.\, Y_{(y)}$

$u_{(x,y)} = Ae^{\frac{k}{2}x}.Be^{-ky}$

Let C=A.B

$u_{(x,y)} = C\, e^{\frac{k}{2}x - ky}$

B.C → $u(x,0) = e^{2x}$ → x=x & y=0

$u(x,0) = e^{2x} = C\, e^{\frac{k}{2}x - 0} = C\, e^{\frac{k}{2}x}$

$e^{2x} = C\, e^{\frac{k}{2}x}$ من التشابه

$C = 1$ & $e^{2x} = e^{\frac{k}{2}x}$

$2 = \frac{k}{2}$ → $\boxed{k=4}$

$u_{(x,y)} = C\, e^{\frac{k}{2}x - ky}$

$u_{(x,y)} = C\, e^{(\frac{4}{2}x - 4y)}$

$u_{(x,y)} = C\, e^{2x - 4y}$

H.W

Solve $u_x + 3\, u_y = 0$, $u(x,0) = 5e^{x}$.

Thank You