SAWA UNIVERSITY

APPLICATION OF PARTIAL DIFFERENTIAL EQUATION (P.D.E)

Sawa University

Engineering Technology

Department of Refrigeration and Air-Conditioning Engineering

$3^{rd} Year$

Lecture 13

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Lecture 13

Application of Partial Differential Equation (P.D.E)

One-dimensional heat flow equation:-

solution of a boundary-value problem by separation of variables. Consider a thin rod length L, with initial temp. f(x) through out and whose ends are held at temp, Zero for all time t > 0 (homog.) , then the temp. $u(x,t)$ in the rod is determined from the boundary – value problem

$$\frac{du}{dt} = C^2 \frac{d^2u}{dx^2} \qquad u(0,t) = 0, \qquad u(L,t) = 0$$

$$u(x,0) = f(x)$$

Solution:

Let $u(x,t) = X(x)T(t)$,

$$\frac{du}{dt} = X\,T'$$

$$\frac{d^2u}{dx^2} = X''\,T$$

$$XT' = C^2\, X''\,T$$

$$\frac{T'}{C^2T} = \frac{X''}{X} = \text{Constant} = k$$

$$k = \begin{Bmatrix} \lambda^2 \\ 0 \\ -\lambda^2 \end{Bmatrix} \qquad \lambda^2 : \textit{real separation constant}$$

note: حالة واحدة فقط هي الصحيحة لذا يجب ان نناقش كل حالة بعد تطبيق المعادلة وال boundary condition

Case I $if\ k = 0$

$$\frac{T'}{C^2T} = \frac{X''}{X} = 0 \quad \rightarrow \quad \frac{T'}{C^2T} = 0 \quad \rightarrow T' = 0$$

$$\int T'\ dt = 0 \qquad \rightarrow T = C$$

$$\& \ \frac{X''}{X} = 0 \quad \rightarrow \quad X'' = 0 \quad \rightarrow \int X''\ dx = 0 \qquad \rightarrow X' = C1$$

$$\int X'\ dx = \int C1 \qquad \rightarrow \qquad x = C1\,x + C2$$

$$u(x,t) = X(x)T(t)$$

$$= (C1\,x + C2)\,C \rightarrow u(x,t) = C3\,x + C4$$

نبحث عن قيم الثوابت C3 ,C4 من ال B.C

B.C

$$u(0,t) = 0 \ \rightarrow \ 0 = 0 + C4 \rightarrow C4 = 0 \ \rightarrow \ u(x,t) = C3\,x$$

$$u(L,t) = 0 \ \rightarrow \ 0 = C3\,L + 0 \rightarrow C3 = 0 \ \rightarrow \ u(x,t) = 0$$

Case II $if\ k = \lambda^2$ (k > 0)

$$\frac{T'}{C^2T} = \frac{X''}{X} = \lambda^2 \quad \rightarrow \quad \frac{T'}{C^2T} = \lambda^2 \quad \rightarrow \int \frac{dT}{T} = \int C^2\,\lambda^2\,dt$$

$$\ln T = C^2\,\lambda^2 t + C1$$

$$T = e^{C^2\,\lambda^2 t + C1} = e^{C^2\,\lambda^2 t}\,.\ e^{C1} \quad \text{Let } A_1 = e^{C1}$$

$$T = A_1\,e^{C^2\,\lambda^2 t}$$

&

$$\frac{X''}{X} = \lambda^2 \ \rightarrow \ X'' - \lambda^2 X = 0 \ \rightarrow \ m^2 - \lambda^2 = 0 \rightarrow m = \pm\lambda$$

$$X(x) = C_1\,e^{\lambda X} + C_2\,e^{-\lambda X}$$

$$u(x,t) = X(x)T(t)$$

$$= (C_1\,e^{\lambda X} + C_2\,e^{-\lambda X})\,A_1\,e^{C^2\,\lambda^2 t}$$

$$u(x,t) = e^{C^2\,\lambda^2 t}\,(A\,e^{\lambda X} + B\,e^{-\lambda X})$$

B.C

$u(0,t)=0 \;\rightarrow\; e^{C^2\lambda^2 t}(A+B)=0 \rightarrow A+B=0 \;\rightarrow\; B=-A$

$u(L,t)=0 \;\rightarrow\; e^{C^2\lambda^2 t}(A\,e^{\lambda L} - A\,e^{-\lambda L})=0$

$\rightarrow A\,e^{C^2\lambda^2 t}(e^{\lambda L} - e^{-\lambda L})=0 \;\rightarrow$

$Either\;\; e^{\lambda L} - e^{-\lambda L}=0 \quad \rightarrow \lambda=0$

Or A=0 → $u(x,t)=0$ → B=0

Case III $if\; k=-\lambda^2$ (k < 0)

$\frac{T'}{C^2 T}=\frac{X''}{X}=-\lambda^2 \;\rightarrow\; \frac{T'}{T}=-C^2\lambda^2 \;\rightarrow \int\frac{dT}{T}=-\int C^2\lambda^2\,dt$

$\ln T = -C^2\lambda^2 t + C1$

$T=e^{-C^2\lambda^2 t+C1}=e^{-C^2\lambda^2 t}.\;e^{C1}$ Let $A_1=e^{C1}$

T= $A_1\,e^{-C^2\lambda^2 t}$

&

$\frac{X''}{X}=-\lambda^2 \;\rightarrow\; X''+\lambda^2 X=0 \;\rightarrow\; m^2+\lambda^2=0 \rightarrow m=\pm\lambda i$

$X(x)=C_1\,Cos\,\lambda x + C_2\,Sin\,\lambda x$

$u(x,t)=X(x)T(t)$

$=(C_1\,Cos\,\lambda x + C_2\,Sin\,\lambda x)A_1\,e^{-C^2\lambda^2 t}$

$u(x,t)=\;e^{-C^2\lambda^2 t}(A\,Cos\,\lambda x + B\,Sin\,\lambda x)$

B.C

$u(0,t)=0 \;\rightarrow\; e^{-C^2\lambda^2 t}(A+0)=0 \rightarrow A\,e^{-C^2\lambda^2 t}=0 \;\rightarrow\; A=0$

$u(L,t)=0 \;\rightarrow\; e^{-C^2\lambda^2 t}(0+B\,Sin\,\lambda L)=0 \rightarrow B=0,\;\; Sin\,\lambda L\neq$

$\rightarrow \lambda L = n\pi$ n= 0,±1, ±2, ±3,…….

$\lambda=\frac{n\pi}{L}$

$u(x,t)=\;e^{-C^2\left(\frac{n^2.\pi^2}{L^2}\right)t} * B\,Sin\,(\frac{n\pi}{L})x$

$$u(x,t)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-C^2(n\pi/L)^2 t}$$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

EX1: For the 1-D heat equation on

$u_t = C^2\, u_{xx}, \quad 0 < x < L, \qquad t > 0\,.$

The ends of the rod are held at zero temperature: $u(0,t)=0, \quad u(L,t)=0 \ for\ t \geq 0$, and the initial temperature along the rod is $u(x,0)=f(x), \quad 0 \leq x \leq L$

Find the temperature distribution $u(x,t)$ using the method of separation of variables.

Sol:

$u(x,t)=X(x)T(t),$

where X depends only on x, and T depends only on t.

$\mathrm{u_t} = \mathrm{X\,T'(t)}, \qquad\qquad \mathrm{u_{xx}} = \mathrm{X''T(t)}.$

$XT' = C^2\, X''\, T$

$\frac{T'}{C^2T}=\frac{X''}{X}=$ Constant $=$k

$$\mathrm{k}=\begin{Bmatrix} \lambda^2 \\ 0 \\ -\lambda^2 \end{Bmatrix} \qquad \lambda^2 : real\ separation\ constant$$

Case I $if\ k=0$

$\frac{T'}{C^2T}=\frac{X''}{X}=0 \quad \rightarrow \quad \frac{T'}{C^2T}=0 \quad \rightarrow T' = 0$

$\int T'\ dt = 0 \qquad \rightarrow T = A$

$\& \frac{X''}{X} = 0 \quad \rightarrow \quad X'' = 0 \quad \rightarrow \quad \int X'' \ dx = 0 \quad \rightarrow X' = B$

$\int X' \ dx = \int B \quad \rightarrow \quad x = B\,x + D$

$u(x,t) = X(x)T(t)$

$= (B\,x + D)\,A$

B.C

$u(0,t) = 0 \quad \rightarrow \quad (B.0 + D)A = 0$ Since A≠0 would give a nontrivial product, we get D= 0

- $u(L,t) = X(L)T(t) = 0 \Rightarrow (BL + 0)A = 0 \Rightarrow B = 0.$

Thus X≡0 and u≡0. So **no nontrivial solution** arises in the k=0 case (except the trivial zero solution).

Case II $if\ k = \lambda^2$ (k > 0)

$\frac{T'}{C^2T} = \frac{X''}{X} = \lambda^2 \quad \rightarrow \quad \frac{T'}{C^2T} = \lambda^2 \quad \rightarrow$ T= $A_1\, e^{C^2\lambda^2 t}$

&

$\frac{X''}{X} = \lambda^2 \rightarrow X'' - \lambda^2 X = 0 \rightarrow \quad m^2 - \lambda^2 = 0 \rightarrow m = \pm\lambda$

$X(x) = B_1\, e^{\lambda X} + B_2\, e^{-\lambda X}$

$u(x,t) = X(x)T(t)$

$u(x,t) = (B_1\, e^{\lambda X} + B_2\, e^{-\lambda X})\, A_1\, e^{C^2\lambda^2 t}$

B.C

$u(0,t) = 0 \rightarrow (B_1 + B_2)\, A_1\, e^{C^2\lambda^2 t} = 0 \rightarrow (B_1 + B_2) = 0 \rightarrow \quad B_2 = -B_1$

$u(L,t) = 0 \rightarrow A_1\, e^{C^2\lambda^2 t}(B_1\, e^{\lambda L} - B_1\, e^{-\lambda L}) = 0$

$\rightarrow A_1 B_1\, e^{C^2\lambda^2 t}(e^{\lambda L} - e^{-\lambda L}) = 0 \quad \rightarrow$

$Either\ e^{\lambda L} - e^{-\lambda L} = 0 \quad \rightarrow \lambda = 0$ but $\lambda > 0$

Or $B_1=0 \rightarrow u(x,t)=0 \quad \rightarrow B_2=0$

$So\ we\ discard\ k > 0$

Case III $if\ k = -\lambda^2$ (k < 0)

$$\frac{T'}{C^2T} = \frac{X''}{X} = -\lambda^2 \quad \rightarrow \quad \frac{T'}{T} = -C^2\lambda^2 \quad \rightarrow \int \frac{dT}{T} = -\int C^2\lambda^2\,dt$$

$$\ln T = -C^2\lambda^2 t + C1$$

$$T = e^{-C^2\lambda^2 t + C1} = e^{-C^2\lambda^2 t} \cdot e^{C1} \quad \text{Let } A = e^{C1}$$

$$T = A\,e^{-C^2\lambda^2 t}$$

&

$$\frac{X''}{X} = -\lambda^2 \rightarrow X'' + \lambda^2 X = 0 \rightarrow m^2 + \lambda^2 = 0 \rightarrow m = \pm\lambda i$$

$$X(x) = B\ Cos\ \lambda x + D\ Sin\ \lambda x$$

$$u(x,t) = X(x)T(t)$$

$$= (B\ Cos\ \lambda x + D\ Sin\ \lambda x)A\,e^{-C^2\lambda^2 t}$$

$$u(x,t) = A\,e^{-C^2\lambda^2 t}(B\ Cos\ \lambda x + D\ Sin\ \lambda x)$$

B.C

$$u(0,t) = 0 \rightarrow A\,e^{-C^2\lambda^2 t}(B.1 + D.0) = 0 \rightarrow B\,e^{-C^2\lambda^2 t} = 0 \rightarrow B = 0$$

$$u(L,t) = 0 \rightarrow A\,e^{-C^2\lambda^2 t}(0 + D\ Sin\ \lambda L) = 0 \rightarrow D Sin\ \lambda L = 0$$

$\rightarrow \lambda L = n\pi$ n= 0,±1, ±2, ±3,…….

$$\lambda = \frac{n\pi}{L}$$

$$u(x,t) = e^{-C^2\left(\frac{n^2.\pi^2}{L^2}\right)t} * A_n\ Sin\ (\frac{n\pi}{L})x$$

Apply the initial condition $u(x,0) = f(x)$:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right),$$

so the Fourier sine coefficients are

$$\boxed{A_n = \frac{2}{L}\int_0^L f(s)\sin\left(\frac{n\pi s}{L}\right)ds, \qquad n = 1, 2, \ldots}$$

Thank You