SAWA UNIVERSITY

SOLVING NONLINEAR EQUATION

Sawa University

Engineering Technology

Department of Refrigeration and Air-Conditioning Engineering

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Lecture 14

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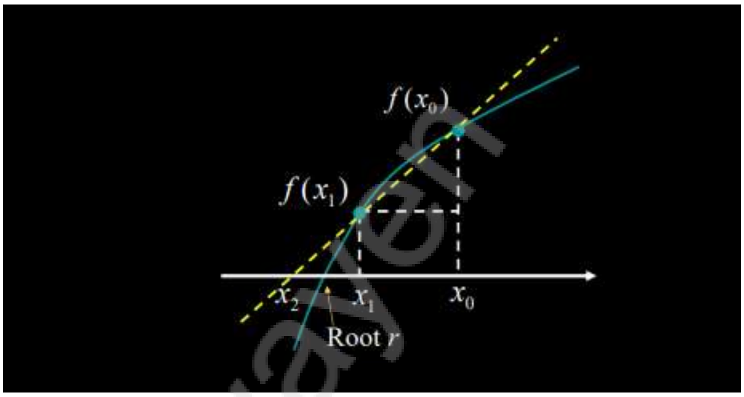
Lecture 14

Solving Nonlinear Equation

Nonlinear Equations



- Given function f, we find value x for which $f\ (x)$ =0
- Solution x is root of equation, or zero of function f
- So problem is known as root finding or zero finding
- Single nonlinear equation in one unknown, where $f: R \to R$

  Solution is scalar x for which $\boldsymbol{f}(\boldsymbol{x})$ = 0
- System of n coupled nonlinear equations in n unknowns, where

$$f: R'' \to R''$$

Solution is vector x for which all components of f are zero simultaneously, $f(x) = 0$

1- Simple Iteration Method

Introduction

In engineering applications (such as heat transfer, fluid flow, and refrigeration systems), we often encounter nonlinear equations of the form:

$$f(x) = 0$$

These equations usually cannot be solved analytically, so we use numerical methods. One of the simplest numerical methods is the Simple Iteration Method, also called the Fixed-Point Iteration Method.

Basic Idea of Simple Iteration

The main idea is to rewrite the nonlinear equation:

$$\boldsymbol{f(x) = 0}$$

into an equivalent form:

$$\boldsymbol{x = g(x)}$$

If $x = \alpha$ is the solution, then it satisfies:

$$\boldsymbol{\alpha = g(\alpha)}$$

Such a value is called a fixed point.

Starting from an initial guess x_0 we generate a sequence:

$$\boldsymbol{x_{n+1}=g(x_n), n = 0, 1, 2, \ldots}$$

If the method converges, the sequence approaches the true solution.

Convergence Condition

The Simple Iteration Method converges if, in a neighborhood of the solution:

$$| g'(x) | < 1$$

- If $| g'(x) | < 1 \rightarrow$ convergent
- If $| g'(x) | \geq 1 \rightarrow$ divergent or unstable

This condition is very important in engineering calculations.

General Algorithm (Steps)

1- Start with a nonlinear equation $f(x) = 0$
2- Rearrange it into the form $x = g(x)$
3- Choose a suitable initial value x_0
4- Compute:

$$\boldsymbol{x_{n+1}=g(x_n),}$$

5- Repeat until:

$$| \boldsymbol{x_{n+1} - x_n} | < \varepsilon$$

where ε varepsilon ε is a small tolerance (10^{-4})

EX1: Use the Simple Iteration Method to find an approximate solution of the nonlinear equation:

$$x^3 - x - 1 = 0$$

starting from x_0=1.

$$x = (x+1)^{1/3}$$

$$g(x) = (x+1)^{1/3}$$

Check convergence

$$g'(x) = \frac{1}{3}(x+1)^{-2/3}$$

For $x \approx 1$:

$$|g'(1)| = \frac{1}{3}(2)^{-2/3} < 1$$

The method is convergent

$$\boldsymbol{x_{n+1} = (x_n + 1)^{1/3}}$$

Iteration 1:

$$x_1 = (x_0+1)^{1/3} = (1+1)^{1/3} = 2^{1/3} \approx 1.2599$$

Iteration 2:

$$x_2 = (x_1+1)^{1/3} = (1.2599+1)^{1/3} = 2.2599^{1/3} \approx 1.3123$$

Iteration 3:

$$x_3 = (x_2+1)^{1/3} = (1.3123+1)^{1/3} = 2.3123^{1/3} \approx 1.3247$$

Iteration 4:

$$x_4 = (x_3+1)^{1/3} = (1.3247+1)^{1/3} = 2.3247^{1/3} \approx 1.3279$$

Iteration	x_n
x_0	1.0000
x_1	1.2599
x_2	1.3123
x_3	1.3247
x_4	1.3279

ملاحظة : نلاحظ أن الفرق بين القيم يصغر كل مرة كما مبين كالاتي :

$$|x_2 - x_1| = |1.3123 - 1.2599| = 0.0524$$

$$|x_3 - x_2| = |1.3247 - 1.3123| = 0.0124$$

$$|x_4 - x_3| = |1.3279 - 1.3247| = 0.0032$$

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$$X = 1.328$$

EX2: Solve the nonlinear equation: $x - cos(x) = 0$ using Simple Iteration Method.

starting from x_0= 0.5

Sol:

$x = cos(x)$

$\boldsymbol{x_{n+1}} = cos(x_n)$

x_0= 0.5

1. $x_1 = \cos(0.5) \approx 0.8776$
2. $x_2 = \cos(0.8776) \approx 0.6390$
3. $x_3 = \cos(0.6390) \approx 0.8027$
4. $x_4 = \cos(0.8027) \approx 0.6948$
5. $x_5 = \cos(0.6948) \approx 0.7682$
6. $x_6 = \cos(0.7682) \approx 0.7192$
7. $x_7 = \cos(0.7192) \approx 0.7524$
8. $x_8 = \cos(0.7524) \approx 0.7301$
9. $x_9 = \cos(0.7301) \approx 0.7451$
10. $x_{10} = \cos(0.7451) \approx 0.7356$
11. $x_{11} = \cos(0.7356) \approx 0.7418$
12. $x_{12} = \cos(0.7418) \approx 0.7372$
13. $x_{13} = \cos(0.7372) \approx 0.7391$

$$X = 0.739$$

This is the approximate solution

2- Newton-Raphson method

Definition (Newton's Method):
The Newton Raphson method is a powerful and well known method for solving equations numerically. It is based on the idea of linear approximation. Usually converges much faster than the linearly convergent methods. Newton method is an iterative method, meaning that it repeatedly attempts to get better approximations to the desired root. Instead of just computing f(x) the derivative $f'(x)$also calculated. it is assumed that the next estimate of the root (solution), is where the tangent crosses the x axis. The values of $f(x)$ and $f'(x)$ are calculated and the process repeated until it converges.It is an open method because initial guess of the root that is needed to get the iterative method started is a single point. While other open methods use two initial guesses of the root but they do not have to bracket the root.

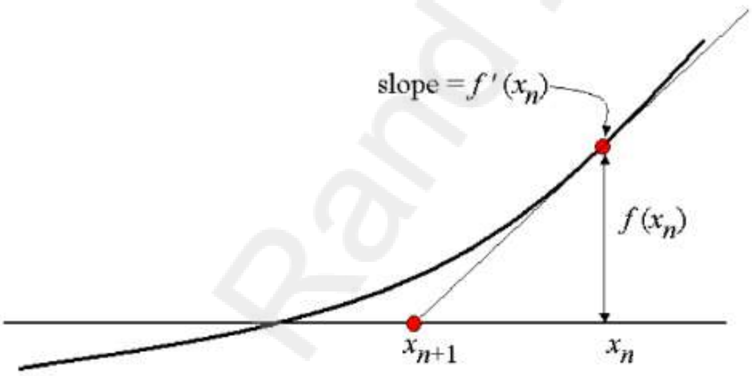

Figure: Newton Raphson method

The Newton-Raphson method is a numerical technique used to find roots of a nonlinear equation:

$$f(x) = 0$$

It is faster than simple iteration and requires the derivative of the function.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where:

- x_n = current approximation
- x_{n+1} = next approximation
- $f'(x_n)$ = derivative of $f(x)$ at x_n

Stopping criterion:

$$|x_{n+1} - x_n| < \text{tolerance}$$

Steps:

1. Choose an initial guess x_0.
2. Compute $f(\boldsymbol{x_n})$ and f $'(\boldsymbol{x_n})$
3. Apply the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

4. Repeat until $|\boldsymbol{x_{n+1}} - \boldsymbol{x_n}|$ is very small

EX3:Use the Newton-Raphson Method to find an approximate root of

$$f(x) = x^2 - 2 = 0$$

starting from x_0=1.

Sol:

$f(x) = x^2 - 2 = 0$

$f'(x) = 2x$

Apply the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

- $x_0 = 1$
- $x_1 = 1 - \frac{1^2-2}{2*1} = 1 + 0.5 = 1.5$
- $x_2 = 1.5 - \frac{1.5^2-2}{2*1.5} = 1.4167$
- $x_3 = 1.4167 - \frac{1.4167^2-2}{2*1.4167} \approx 1.4142$
- $x_4 = 1.4142 - \frac{1.4142^2-2}{2*1.4142} \approx 1.4142$

$|x_4 - x_3| \approx 0.0000 < 0.001$

الفرق بين اخر قيمتين قيمة صغيرة جدا اذا بامكاننا التوقف

$$X = 1.414$$

This is the approximate solution

EX4:Use Newton's Method to approximate the cube root of 5. Choose x_0=2.

Sol: The cube root of 5 is the solution to the equation: $x^3 = 5 \rightarrow x=\sqrt[3]{5}$

$x^3 = 5 \rightarrow f(x) = x^3 - 5 = 0$

$f'(x) = 3x^2$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 5}{3x_n^2}$$

x_0=2

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{2^3 - 5}{3 \cdot 2^2} = 2 - \frac{8-5}{12} = 2 - 0.25 = 1.75$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.75 - \frac{1.75^3 - 5}{3 \cdot 1.75^2} \approx 1.71088435$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 1.71088435 - \frac{1.71088435^3 - 5}{3 \cdot (1.71088435)^2} \approx 1.70997642$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 1.70997642 - \frac{1.70997642^3 - 5}{3 \cdot (1.70997642)^2} \approx 1.70997594$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \approx 1.70997594 - \frac{(1.70997594)^3 - 5}{3 \cdot (1.70997594)^2} \approx 1.70997594$$

At this point, the difference between $x5$ and $x4$ is extremely small.

$$\boxed{\sqrt[3]{5} \approx 1.70997594}$$

H.W: Solve $x^4 - x - 7 = 0$ by the Newton-Raphson method, correct up to 6 significant digits. Let $x_0 = 1.5$

3-**Finite Difference Methods Overview**

Finite difference methods approximate derivatives by using discrete points. They are commonly used for solving differential equations numerically, but they can also be applied to nonlinear equations in various contexts.

In finite difference methods, we replace derivatives with difference quotients. For example, the first derivative of a function $f(x)$ at a point x can be approximated as:

- **Forward difference:**

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

- **Backward difference:**

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}$$

- **Central difference:**

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Here, h is a small step size.

Summary of Method:

1. Start with initial guess x_0
2. Compute $f(x_n)$
3. Approximate derivative using finite differences:

$$f'(x_n) \approx \frac{f(x_n+h) - f(x_n)}{h}$$

4. Update using Newton-Raphson formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

5. Repeat until convergence

Ex5:

Solve:

$$f(x) = x^2 - 2 = 0$$

- The solution is obviously $\sqrt{2} \approx 1.4142$, but we'll solve it numerically.
- Initial guess: $x_0 = 1$
- Step size for finite difference: $h = 0.001$

: **Newton-Raphson formula with finite difference**

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \approx \frac{f(x_n + h) - f(x_n)}{h}$$

1-

$$f(x_0) = f(1) = 1^2 - 2 = -1$$

$$f(x_0 + h) = f(1.001) = (1.001)^2 - 2 \approx 1.002001 - 2 = -0.997999$$

$$f'(x_0) \approx \frac{-0.997999 - (-1)}{0.001} = \frac{0.002001}{0.001} = 2.001$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{-1}{2.001} \approx 1 + 0.4998 = 1.4998$$

2-

$$f(x_1) = (1.4998)^2 - 2 \approx 2.2494 - 2 = 0.2494$$

$$f(x_1 + h) = (1.5008)^2 - 2 \approx 2.2524 - 2 = 0.2524$$

$$f'(x_1) \approx \frac{0.2524 - 0.2494}{0.001} = 3.0$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.4998 - \frac{0.2494}{3.0} \approx 1.4998 - 0.0831 = 1.4167$$

3-

$$f(x_2) = (1.4167)^2 - 2 \approx 2.0069 - 2 = 0.0069$$

$$f(x_2 + h) = (1.4177)^2 - 2 \approx 2.0100 - 2 = 0.0100$$

$$f'(x_2) \approx \frac{0.0100 - 0.0069}{0.001} = 3.1$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.4167 - \frac{0.0069}{3.1} \approx 1.4145$$

4-

$$f(x_3) \approx (1.4145)^2 - 2 \approx 0.0007$$

$$f'(x_3) \approx \frac{f(1.4155) - f(1.4145)}{0.001} \approx 2.83$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx 1.4145 - \frac{0.0007}{2.83} \approx 1.4142$$

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$$\boxed{x \approx 1.4142}$$

Thank You