SAWA UNIVERSITY

SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS DIRECT AND INDIRECT METHODS

Sawa University

Engineering Technology

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Lecture 15

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Solution of simultaneous linear equations, Direct and Indirect methods

Introduction

The basic idea for this method is **finding the value of (x)** by arranging several simulation equations in a manner that simplify the solution.

A simple example can explain the above text:

Suppose we have several equations that contain the unknown value of (x), we cannot find the value of (x_s) from one equation, like

$\boldsymbol{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1}$

Where

The values of $\boldsymbol{x_1 \to x_n}$ is unknown

$\boldsymbol{a_{11} \to a_{1n}}$ Constant

$\boldsymbol{b_1}$ Constant

So:

If we want to know the values of (x), we should have several equations having a number equal to the number of unknown,

$$\begin{array}{c}
\boldsymbol{a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1} \\
\boldsymbol{a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2} \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\boldsymbol{a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m}
\end{array}$$

Simultaneous linear equations are a set of linear equations that contain the same unknown variables. The objective is to find the values of these variables that satisfy all equations at the same time.

General form:

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

Matrix form:

$$AX = B$$

These equations appear frequently in engineering applications such as heat balance, mass flow analysis, and refrigeration system modeling

Classification of Solution Methods

1. Direct Methods

Direct methods give the exact solution (neglecting rounding errors) after a finite number of steps. (**Gauss Elimination, Gauss Jordan**)

2. Indirect (Iterative) Methods

Approximate solutions using repeated iterations **(Gauss-Seidel, Jacobi)**

1. Direct Methods (Exact Method)
A- Gauss Elimination Method

Engineers often need to solve large systems of linear equations; for example in determining the forces in a large framework or finding currents in a complicated electrical circuit. The method of Gauss elimination provides a systematic approach to their solution. The technique involves combining equations in order to eliminate unknown, although it is one of the earliest methods for solving simultaneous equations, it remains among the most important algorithms in use today

المهندسون غالبًا يحتاجون إلى حل أنظمة كبيرة من المعادلات الخطية؛ على سبيل المثال، عند تحديد القوى في هيكل كبير أو إيجاد التيارات في دائرة كهربائية معقدة. توفر طريقة الحذف بالجوس (Gauss Elimination) نهجًا منظمًا لحل هذه الأنظمة.

Principle: Transform the system to an upper triangular form and solve by back substitution.

Steps: الخطوات

1. Write the augmented matrix. كتابة المصفوفة الموسعة
2. Eliminate variables step by step.التخلص من المتغيرات خطوة بخطوة
3. Solve from the last equation upwards. الحل بدءًا من المعادلة الأخيرة إلى الأعلى

EX1: Consider we have three simultaneous equations

$$2x_1 + x_2 = 7$$
$$x_1 + x_2 = 5$$

1. **Eliminate x_1 from $r2$**

- Multiply r2 by 2 and subtract r1:

$$2(x_1 + x_2) - (2x_1 + x_2) = 10 - 7$$

$$x_2 = 3$$

2. **Back Substitution**

Substitute $x_2 = 3$ into r2:

$$x_1 + 3 = 5 \Rightarrow x_1 = 2$$

$$x_1 = 2, \quad x_2 = 3$$

EX2: Solve the system:

$$r_1: \quad x_1 + x_2 + x_3 = 6$$
$$r_2: \quad 2x_1 + x_2 + x_3 = 9$$
$$r_3: \quad x_1 + 2x_2 + x_3 = 8$$

Sol: 1-

Eliminate x_1 from r2:

- Subtract r1 from r2:

$$(2x_1 + x_2 + x_3) - (x_1 + x_2 + x_3) = 9 - 6$$

$$x_1 + 0 + 0 = 3 \quad \Rightarrow x_1 = 3$$

2- Find x_2 and x_3

- Substitute $x_1 = 3$ into r1:

$$3 + x_2 + x_3 = 6 \quad \Rightarrow x_2 + x_3 = 3 \quad (1)$$

- Substitute $x_1 = 3$ into r3:

$$3 + 2x_2 + x_3 = 8 \quad \Rightarrow 2x_2 + x_3 = 5 \quad (2)$$

3-

- From (1): $x_3 = 3 - x_2$
- Substitute into (2):

$$2x_2 + (3 - x_2) = 5 \quad \Rightarrow x_2 + 3 = 5 \quad \Rightarrow x_2 = 2$$

- Then $x_3 = 3 - 2 = 1$

$$\boxed{x_1 = 3, \quad x_2 = 2, \quad x_3 = 1}$$

H.W (1)

Solve the following sets of linear equations using Gauss-Elimination method.

$$x_1 + x_2 - x_3 = 0$$
$$3x_1 - x_2 + x_3 = 4$$
$$x_1 - 2x_2 + 2x_3 = 3$$

B- Gauss- Jordan Method

The Gauss-Jordan method is an extension of Gaussian elimination. The goal is to reduce the augmented matrix of a system of linear equations directly into **reduced row-echelon form** also called the **identity matrix form**, so that the solution can be read directly **without back substitution**.

- هذه الطريقة تستخدم لتحويل **المصفوفة الموسعة (augmented matrix)** لنظام المعادلات إلى **شكل المصفوفة المثلثية الكاملة أو مصفوفة الوحدةZ(Identity Matrix)**

- عندما نصبح المصفوفة بالشكل هذا، يمكننا قراءة الحل مباشرة **بدون الحاجة للرجوع للخلف** كما في Gaussian elimination.

خطوات Steps

1. اكتب المصفوفة الموسعة [A|B] لنظام المعادلات.
2. اجعل الرقم الأساسي في الصف الأول = 1 إذا لم يكن كذلك.
3. اجعل كل العناصر الأخرى في عمود هذا = 0 سواء فوقه أو تحته
4. انتقل للصف/العمود التالي وكرر الخطوتين 2 و3 حتى تصبح المصفوفة Identity.
5. العمود الأخير الآن يحتوي على الحل مباشرة

EX3: Solve

$$x_1 + x_2 = 5$$
$$2x_1 + 3x_2 = 11$$

$$\begin{bmatrix} 1 & 1 & | & 5 \\ 2 & 3 & | & 11 \end{bmatrix}$$

- Eliminate first column in second row

R2 → R2 - 2*R1

$$\begin{bmatrix} 1 & 1 & | & 5 \\ 0 & 1 & | & 1 \end{bmatrix}$$

- Eliminate first column in first row above pivot in second row

R1 → R1 - 1*R2

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$\boxed{x_1 = 4, \quad x_2 = 1}$$

2. Indirect (Iterative) Methods

Gauss-Seidel Iteration Method

Iterative technique method is one in which we start from an approximate and end up with the true solution as the number of iterations becomes large.

Assume the equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The solution for the new values is:

$$x_1^{(k+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)}$$

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k+1)} - \frac{a_{23}}{a_{22}} x_3^{(k)}$$

$$x_3^{(k+1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_2^{(k+1)} - \frac{a_{32}}{a_{33}} x_3^{(k+1)}$$

In General

$$x_1^{(k+1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(k)} - \frac{a_{13}}{a_{11}} x_3^{(k)} \dots \dots - \frac{a_{1n}}{a_{11}} x_n^{(k)}$$

$$x_2^{(k+1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(k+1)} - \frac{a_{23}}{a_{22}} x_3^{(k)} \dots \dots - \frac{a_{2n}}{a_{22}} x_n^{(k)}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$x_n^{(k+1)} = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1^{(k+1)} - \frac{a_{n2}}{a_{nn}} x_2^{(k+1)} \dots \dots - \frac{a_{n\,n-1}}{a_{nn}} x_{n-1}^{(k+1)}$$

Important Note

The **first step** for solving any **gauss-Seidel** problem is to apply the condition below

$$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$$

EX4: Solve

$$4x_1 - x_2 = 3$$
$$-2x_1 + 5x_2 = -1$$

Sol:

$$x_1 = \frac{3 + x_2}{4}, \quad x_2 = \frac{-1 + 2x_1}{5}$$

- Initial guess

$$x_1^{(0)} = 0, \quad x_2^{(0)} = 0$$

- Iteration 1

$$x_1^{(1)} = \frac{3+0}{4} = 0.75$$

$$x_2^{(1)} = \frac{-1+2(0.75)}{5} = \frac{-1+1.5}{5} = 0.1$$

- Iteration 2

$$x_1^{(2)} = \frac{3+0.1}{4} = 0.775$$

$$x_2^{(2)} = \frac{-1+2(0.775)}{5} = \frac{-1+1.55}{5} = 0.11$$

Thank You